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THE MOTION OF AN ABSOLUTELY RIGID BODY ON A TWO-DEGREES-OF-FREEDOM JOINT IN A UNIFORM GRAVITATIONAL FIELD†

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The motion of an absolutely rigid body attached to a fixed base by a two-degrees-of-freedom joint in a uniform gravitational field parallel to the fixed axis of the joint is studied qualitatively. Various kinds of motion are described and analysed, depending on the total mechanical energy and the projection of the angular momentum of the body onto the fixed axis of the joint as well as on the inertial parameters of the system.

This paper is a continuation of [1].

1. EQUATION OF MOTION

We consider an absolutely rigid body attached to a fixed base by a two-degrees-of-freedom joint with axes perpendicular to one another (Fig. 1). The joint is assumed to be ideal, i.e. no friction is taken into account on any of its axes. To describe the motion we introduce two right-handed Cartesian systems of coordinates: a fixed (inertial) system $X_1X_2X_3$ and a system $x_1x_2x_3$ attached to the rigid body. We place the origin of each system of coordinates at the point of intersection O of the axes of the joint, the axes X_3 and x_1 being directed along the fixed and the moving axes of the joint, respectively. All kinematically feasible states of the rigid body (systems of coordinates $x_1x_2x_3$) relative to the system of coordinates $X_1X_2X_3$ can be described by two angles: the angle α between the axes X_1 and x_1 and the angle β between the x_2 axis and the plane X_1X_2. Henceforth these angles will be taken as the generalized coordinates of the mechanical system under consideration.

The matrix Γ of the transformation from the system of coordinates $X_1X_2X_3$ to $x_1x_2x_3$ can be expressed in terms of α and β as follows [1]:

$$\Gamma = \begin{Vmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha\cos\beta & \cos\alpha\cos\beta & \sin\beta \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{Vmatrix} \tag{1.1}$$

The kinetic energy of an absolutely rigid body on a two-degrees-of-freedom joint is equal to [1]

$$T = \tfrac{1}{2}K(\beta)\dot{\alpha}^2 + \tfrac{1}{2}J_{11}\dot{\beta}^2 - b(\beta)\dot{\alpha}\dot{\beta} \tag{1.2}$$

$$K(\beta) = J_{22}\sin^2\beta + J_{33}\cos^2\beta - 2J_{23}\sin\beta\cos\beta, \quad b(\beta) = J_{12}\sin\beta + J_{13}\cos\beta$$

Here J_{ii} ($i = 1, 2, 3$) are the axial moments of inertia and $J_{ij} = J_{ji}$ ($i \neq j, i, j = 1, 2, 3$) are the products of inertia of the rigid body in the system of coordinates $x_1x_2x_3$. Henceforth it is assumed that the ellipsoid of inertia of the rigid body is non-degenerate and the corresponding inertia tensor $J = \| J_{ij} \|$ is positive definite.

Let the mechanical system be subject to a uniform gravitational field with intensity vector $\mathbf{g}$. In this case the potential energy of the rigid body can be expressed by the scalar product

$$U = -m(\mathbf{g}, \mathbf{r}_c) \tag{1.3}$$

where $\mathbf{r}_c$ is the position vector of the centre of mass relative to O and m is the mass of the body.

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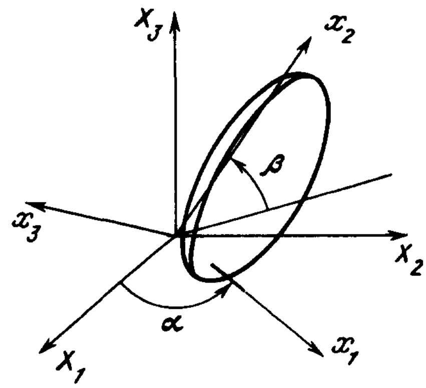

Fig. 1.

We denote by g_1, g_2, g_3 the components of $\mathbf{g}$ in the fixed system of coordinates $X_1X_2X_3$ and by ρ_1, ρ_2, ρ_3, the components of $\mathbf{r}_c$ in the moving system of coordinates $x_1x_2x_3$. Taking the projections of $\mathbf{g}$ onto the axes of the system of coordinates $x_1x_2x_3$, using (1.1), and expanding the scalar product (1.3), we get

$$U = -m[\rho_1(g_1\cos\alpha + g_2\sin\alpha) + \rho_2(-g_1\sin\alpha\cos\beta + g_2\cos\alpha\cos\beta + g_3\sin\beta) + \\ +\rho_3(g_1\sin\alpha\sin\beta - g_2\cos\alpha\sin\beta + g_3\cos\beta)] \tag{1.4}$$

The Lagrange equations of a mechanical system with kinetic energy (1.2) and potential energy (1.4) have the form

$$K(\beta)\ddot{\alpha} - b(\beta)\ddot{\beta} + [(J_{22} - J_{33})\sin 2\beta - 2J_{23}\cos 2\beta]\dot{\alpha}\dot{\beta} - (J_{12}\cos\beta - J_{13}\sin\beta)\dot{\beta}^2 = \tag{1.5}$$
$$= m[\rho_1(-g_1\sin\alpha + g_2\cos\alpha) + (\rho_3\sin\beta - \rho_2\cos\beta)(g_1\cos\alpha + g_2\sin\alpha)]$$
$$-b(\beta)\ddot{\beta} + J_{11}\ddot{\beta} - \tfrac{1}{2}[(J_{22} - J_{33})\sin 2\beta - 2J_{23}\cos 2\beta]\dot{\alpha}^2 =$$

An absolutely rigid body on an ideal two-degrees-of-freedom joint subject to a uniform gravitational field is a conservative mechanical system, and hence the total energy $E = T + U$ is a first integral of the equations of motion (1.5).

The equations of motion (1.5) form a fourth-order system of non-linear differential equations containing a large number of parameters, which makes it practically impossible to study the system in the general case. However, in a number of special cases, considered in the following sections, the system can be significantly simplified and admits of an effective qualitative analysis.

2. THE CASE OF GRAVITATIONAL FIELD PARALLEL TO THE FIXED AXIS OF THE JOINT

If the gravitational field is parallel to the fixed X_3 axis of the joint, then (1.4) can be simplified, and takes the form

$$U = mg(\rho_2\sin\beta + \rho_3\cos\beta) \quad (g = -g_3) \tag{2.1}$$

Henceforth we shall assume that $g > 0$. This involves no loss of generality, for if the gravitational field is parallel to the fixed axis of the joint, the system of coordinates $X_1X_2X_3$ can always be chosen in such a way that the X_3 axis is directed opposite to the vector $\mathbf{g}$, so that $g > 0$.

The potential energy (2.1) is independent of α. Since the kinetic energy (1.2) is also independent of α, it follows that α is the cyclic generalized coordinate when $g_1 = g_2 = 0$, which implies that

$$L = \partial T / \partial\dot{\alpha} = K(\beta)\dot{\alpha} - b(\beta)\dot{\beta} \tag{2.2}$$

is a first integral of the system under consideration. L is the projection of the angular momentum onto the fixed X_3 axis.

The presence of the first two integrals, $E = T + U$ and L, enables us to separate the variables α and β and reduce the integration of (1.5) to quadratures. Expressing $\dot\alpha$ in terms of β and $\dot\beta$ and using (2.2), we have

$$\dot\alpha = [L + b(\beta)\dot\beta] / K(\beta) \tag{2.3}$$

Substituting (2.3) into (1.2), the energy integral can be represented as follows:

$$\tfrac{1}{2}a(\beta)\dot\beta^2 + \Pi(\beta, L) = E, \quad a(\beta) = [J_{11}K(\beta) - b^2(\beta)] / K(\beta) \tag{2.4}$$

$$\Pi(\beta, L) = L^2 / (2K(\beta)) + mg(\rho_2 \sin\beta + \rho_3 \cos\beta)$$

Note that $K(\beta)$ and $a(\beta)$ in (2.3) and (2.4) are positive because the inertia tensor of a rigid body is positive definite [1].

Equation (2.4) describes the variation of β and can be reduced to quadratures

$$I(\beta_0, \beta) = \pm(t - t_0), \quad I(x, y) = \int_x^y \left\{\frac{a(\xi)}{2[E - \Pi(\xi, L)]}\right\}^{1/2} d\xi \tag{2.5}$$

where $\beta_0 = \beta(t_0)$ and t_0 is the initial instant of time. The sign on the right-hand side of (2.5) is the same as that of $\dot\beta$ when $\dot\beta \neq 0$ or that of $-\partial\Pi/\partial\beta$ when $\dot\beta = 0$ and $\partial\Pi/\partial\beta \neq 0$. If $\dot\beta = 0$ and $\partial\Pi/\partial\beta = 0$, the system is in a steady state with respect to β.

The function $\beta = \beta(t)$ is defined implicitly by (2.5). Substituting this function into (2.3), we can find $\alpha = \alpha(t)$ by integrating the right-hand side of (2.3) with respect to t.

We will investigate qualitatively all possible kinds of motion of system (2.3), (2.4). First we shall study the motion with respect to β described by Eq. (2.4). This equation is formally identical with the equation of motion of a mechanical system with one degree of freedom and with kinetic and potential energy—$a(\beta)\dot\beta^2/2$ and $\Pi(\beta, L)$, respectively. The phase plane method serves as the most convenient and graphic method of studying such systems. It consists of constructing the graphs of $\dot\beta(\beta)$ from (2.4) for various values of E and determining from these graphs the characteristic features of the motion, which depend on the form of $\Pi(\beta, L)$.

3. MOTION WITH RESPECT TO β FOR $g_1 = g_2 = 0$. SPECIAL CASES

3.1. $\rho_2 = \rho_3 = 0$. This means that the centre of mass of the rigid body lies on the moving x_1 axis of the joint. In this case the gravitational force does not affect the motion of the system at all, the behaviour of the system being the same as the inertial motion of a rigid body on a two-degrees-of-freedom joint studied in [1].

3.2. $J_{22} = J_{33}, J_{23} = 0$. In this case $K(\beta) = J_{22} =$ const and $\Pi(\beta, L)$ is a sinusoid $mg(\rho_2 \sin\beta + \rho_3 \cos\beta)$ of period 2π shifted along the ordinate axis by an amount $\Lambda = L^2/(2J_{22})$. This function has two extrema in the interval $[0, 2\pi)$: a maximum $\Pi_{max} = \Lambda + \mu$ ($\mu = mg(\rho_2^2 + \rho_3^2)^{1/2}$ which is reached at

$$\beta = \beta_1 = \begin{cases} \beta_*, & \text{if} \quad \rho_2 \geq 0 \\ 2\pi - \beta_*, & \text{if} \quad \rho_2 < 0 \end{cases}, \quad \beta_* = \arccos\frac{\rho_3}{(\rho_2^2 + \rho_3^2)^{1/2}} \tag{3.1}$$

and a minimum $\Pi_{min} = \Lambda - \mu$ which is reached at

$$\beta = \beta_2 = \begin{cases} \beta_* + \pi, & \text{if} \quad \rho_2 \geq 0 \\ \pi - \beta_*, & \text{if} \quad \rho_2 < 0 \end{cases} \tag{3.2}$$

In the case under consideration the phase plane of system (2.4) is qualitatively the same as that of a mathematical or physical pendulum. The fact that the "moment of inertia" $a(\beta)$ of a pendulum depends on β is unimportant in a qualitative analysis.

This implies the following possibilities for the motion with respect to β:

1. If $E < \Pi_{min}$, no motion is possible.

2. $\Pi_{min} < E < \Pi_{max}$, the system performs periodic oscillations about the stable steady state $\beta = \beta_2$ corresponding to a minimum of $\Pi(\beta, L)$. The angle β varies in the range $\beta_- \leqslant \beta \leqslant \beta_+$, where

$$\beta_\pm = \beta_2 \pm A, \quad A = \arccos\{[L^2/(2J_{22}) - E]/[mg(\rho_2^2 + \rho_3^2)^{1/2}]\} \tag{3.3}$$

The amplitude and period of these oscillations are equal, respectively, to A and

$$\tau_\upsilon = 2I(\beta_-, \beta_+) \tag{3.4}$$

Formula (3.3) can be obtained by solving the equation $\Lambda + mg(\rho_2 \sin\beta + \rho_3 \cos\beta) = E$.

3. If $E > \Pi_{max}$, the system rotates periodically with period

$$\tau_r = I(0, 2\pi) \tag{3.5}$$

4. If $E > \Pi_{min}$, the system is in the stable equilibrium state $\beta = \beta_2$.

5. If $E > \Pi_{max}$, the system is either in the unstable equilibrium state $\beta = \beta_1$, or the body rotates about the x_1 axis of the moving joint with infinite period. In the phase plane this corresponds to the motion along the separatrix.

4. MOTION WITH RESPECT TO β FOR $g_1 = g_2 = 0$. THE GENERAL CASE

In the general case the function $\Pi(\beta, L)$ in (2.4) can be represented as (see also [1])

$$\begin{aligned} &\Pi(\beta, L) = L^2/[J_{22} + J_{33} + R\cos(2\beta + \nu_1)] + \mu\cos(\beta + \nu_2) \\ &R = [(J_{33} - J_{22})^2 + 4J_{23}^2]^{1/2} \\ &\cos\nu_1 = (J_{33} - J_{22})/R, \quad \sin\nu_1 = 2J_{23}/R \\ &\cos\nu_2 = mg\rho_3/\mu, \quad \sin\nu_2 = -mg\rho_2/\mu \end{aligned} \tag{4.1}$$

When analysing (4.1) it is convenient to rewrite it as follows:

$$\begin{aligned} &\Pi(\beta, L) = \mu f(x), \quad f(x) = \varepsilon/(1 + \lambda\cos 2x) + \cos(x + \nu) \\ &\varepsilon = L^2/[\mu(J_{22} + J_{33})], \quad \lambda = R/(J_{22} + J_{33}) \\ &x = \beta + \nu_1/2, \quad \nu = \nu_2 - \nu_1/2 \end{aligned} \tag{4.2}$$

Since the inertia tensor $J = \| J_{ij} \|$ $(i, j = 1, 2, 3)$ is positive definite, it follows that $J_{22} > 0, J_{22}J_{33} - J_{23}^2 > 0$, which implies a limit for λ: $0 \leqslant \lambda < 1$. The case $\lambda = 0$ corresponds to $J_{22} = J_{23}$, $J_{23} = 0$ and was considered in Section 3. In Section 4 we therefore assume that $0 < \lambda < 1$.

The qualitative nature of the motion of the system with respect to β depends on the form of $\Pi(\beta, L)$. Each stationary point of this function corresponds to an equilibrium state with respect to β, the type of which is determined by the type of stationary point. A minimum of $\Pi(\beta, L)$ corresponds to a stable equilibrium state of the centre type, a maximum corresponds to an equilibrium state of the saddle type, and a point of inflection corresponds to an unstable equilibrium such that arbitrarily small deviations from it give rise to finite displacements in only one specific direction.

One can see from (4.2) that the dependence of $\Pi(\beta, L)$ on β is determined by $f(x)$, which is a 2π-periodic function of x and ν. Henceforth we shall assume without loss of generality that $0 \leqslant x \leqslant 2\pi$ and $0 \leqslant \nu \leqslant 2\pi$.

We shall find the number of stationary points of the function (4.2) depending on ε, ν and λ. Differentiating $f(x)$, we obtain the equation

$$2\varepsilon\lambda \sin 2x/(1 + \lambda\cos 2x)^2 - \sin(x + \nu) = 0 \tag{4.3}$$

whose roots are the desired stationary points. In general, this equation does not admit of a simple analytical solution expressing all roots as functions of ε, ν, λ. Thus, we express ε in terms of x, λ, ν using Eq. (4.3)

$$\varepsilon = \varepsilon(x;\ \nu, \lambda) = (1 + \lambda \cos 2x)^2 \sin(x + \nu)/(2\lambda \sin 2x) \tag{4.4}$$

$$x \neq \pi i/2 \quad (i = 0, 1, 2, 3)$$

and study (4.4) as a function of x for various λ and ν. The total preimage of a fixed value ε taken in the interval $[0, 2\pi)$, which corresponds to (4.4), is the set of stationary points of $f(x)$ for the given ε if $x = \pi i/2$ $(i = 0, 1, 2, 3)$ are not the roots of Eq. (4.3). These values are the roots of Eq. (4.3) only if $\nu = \pi i/2$ $(i = 0, 1, 2, 3)$. In this case the corresponding numbers of the form $x = \pi i/2$ must be added to the set of stationary points found from (4.4).

From (4.4) it follows that $\varepsilon(x;\ \nu, \lambda) = -\varepsilon(x + \pi;\ \nu, \lambda)$ and $\varepsilon(x;\ \nu, \lambda) = -\varepsilon(x;\ \nu + \pi, \lambda)$. This enables us to carry out the computations in the subset $0 \leqslant x < \pi$, $0 \leqslant \nu \leqslant \pi$, rather than in the whole set $0 \leqslant x < 2\pi$, $0 \leqslant \nu \leqslant 2\pi$, without loss of generality.

For an arbitrary $\nu \in [0, 2\pi)$ an analytical study of (4.4) proves difficult. Below we present detailed results applying to some special cases ($\nu = 0$ and $\nu = \pi/2$) which admit of such a study, and we present the graphs of (4.4) constructed for a number of various values of ν and λ.

4.1. $\nu = 0$. In this case, for any ε, Eq. (4.3) has roots $x = 0$ and $x = \pi$, which cannot be found from (4.4). Analysis indicates that $x = 0$ corresponds to a maximum of the function $f(x)$ in (4.2) if $\varepsilon < \varepsilon_* = (1 + \lambda)^2/(4\lambda)$ and a minimum if $\varepsilon > \varepsilon_*$. If $\varepsilon = \varepsilon_*$, then $f(x)$ has a maximum at $x = 0$ when $\lambda \leqslant 1/7$ and $\varepsilon < \varepsilon_*$ and a minimum when $\lambda > 1/7$. Since $f(x)$ changes into $-f(x)$ when x is replaced by $x + \pi$ and ε by $-\varepsilon$, the properties of the stationary point $x = 0$ imply that $f(x)$ has a maximum at $x = \pi$ if $\varepsilon < -\varepsilon_*$ and a minimum if $\varepsilon < -\varepsilon_*$. If $\varepsilon = -\varepsilon_*$, then $x = \pi$ is a minimum of $f(x)$ when $\lambda \leqslant 1/7$ and a maximum when $\lambda > 1/7$.

In Figs 2 and 3 the bold solid lines represent graphs of $\varepsilon(x, 0, \lambda)$, which characterize qualitatively the behaviour of $\varepsilon(x;\ 0, \lambda)$ for $\lambda \leqslant 1/7$ and $\lambda > 1/7$, respectively, as well as the vertical lines $x = 0$ and $x = \pi$ corresponding to the stationary points of $f(x)$ from (4.2) for $\nu = 0$ and any ε. We observe that for $\lambda > 1/7$ the function $\varepsilon(x;\ 0, \lambda)$ has local minima at $x = x_* = \arccos[(1 - \lambda)/(6\lambda)]$ and $x = 2\pi - x_*$ with value $\varepsilon = \varepsilon_0 = 2[2(1 - \lambda)/3]^{3/2}\lambda^{-1/2}$ and local maxima at $x = \pi \pm x_*$ with value $\varepsilon = -\varepsilon_0$. Some sections of the solid lines are accompanied by dashed lines. The function $f(x)$ has a minimum at each stationary point that corresponds to these sections and a maximum at each of the remaining points when $\varepsilon \neq \varepsilon_0$. When $\varepsilon = \varepsilon_0$, the function $f(x)$ has a point of inflection.

Diagrams similar to those in Figs 2 and 3 enable us to determine the number and type of stationary points of $f(x)$ for any given ε and, consequently, also the number and type of the corresponding equilibrium states of the mechanical system in hand with respect to β. To this end one must draw the straight line $\varepsilon = \text{const}$. Each point of intersection of this line and the bold solid lines in the diagram corresponds to a stationary point, the type of which depends on whether or not a dashed line is present next to the solid one in a neighbourhood of the point of intersection. In particular, these diagrams imply that the number of stationary points may be equal to two, four, or six, depending on λ and ε.

As an illustration, in Fig. 4 we present the graphs of $f(x)$ in the interval $0 \leqslant x \leqslant \pi$ for $\nu = 0$, $\lambda = 3/4$ and various values of ε: $1 - \varepsilon = 0.1$ ($\varepsilon < \varepsilon_0 = \sqrt{(2/9)}$; $2 - \varepsilon = \varepsilon_0$; $3 - \varepsilon = 1/3$ ($\varepsilon_0 < \varepsilon < \varepsilon_* = 49/48$); $4 - \varepsilon = 1.5$ ($\varepsilon > \varepsilon_*$). The graphs corresponding to the interval $\pi < x < 2\pi$ can be obtained by taking the mirror image of the curves in Fig. 4 with respect to the straight line $x = \pi$.

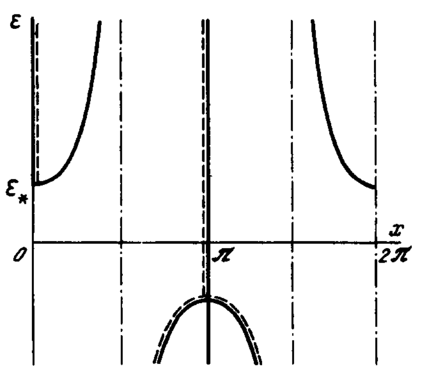

Fig. 2.

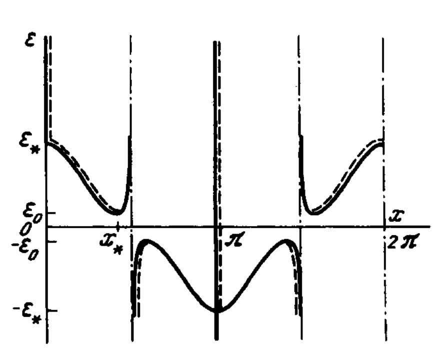

Fig. 3.

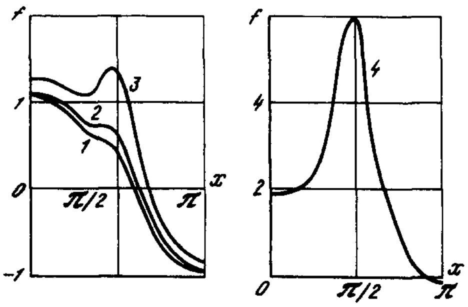

Fig. 4.

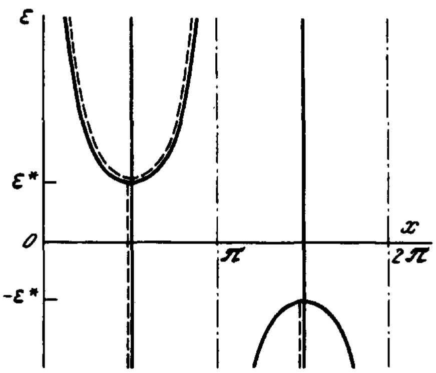

Fig. 5.

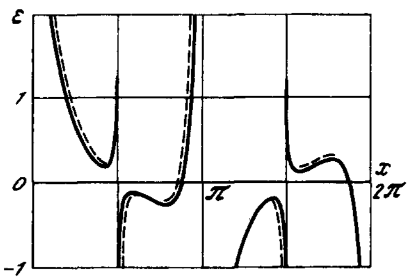

Fig. 6.

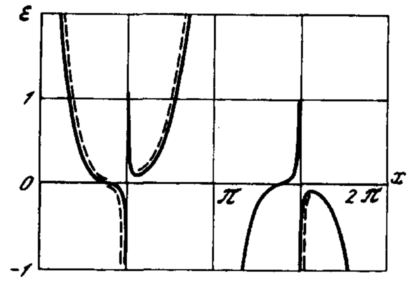

Fig. 7.

4.2. $\nu = \pi/2$. In this case, for any ε, Eq. (4.3) has roots $x = \pi/2$ and $x = 3\pi/2$ which cannot be found from (4.4). The function $f(x)$ in (4.2) has a minimum at $x = \pi/2$ if $\varepsilon \leqslant \varepsilon^* = (1-\lambda)^2/(4\lambda)$ and a maximum if $\varepsilon > \varepsilon^*$. The point $x = 3\pi/2$ is a maximum when $\varepsilon \geqslant -\varepsilon^*$ and a minimum when $\varepsilon < -\varepsilon^*$.

The diagram used to analyse the stationary points of the function (4.2) for $\nu = \pi/2$ is presented in Fig. 5, from which one can see that in this case $f(x)$ has either two or four stationary points.

4.3. Arbitrary $\nu \neq \pi/2$. In this case $\varepsilon(x;\, \nu, \lambda)$ defined by (4.4) vanishes at $x = \pi - \nu$ and $x = 2\pi - \nu$ if $\nu \in (0, \pi)$ and at $x = 2\pi - \nu$ and $x = 3\pi - \nu$ if $\nu \in (\pi, 2\pi)$. As $x \to +0, x \to \pi/2 \pm 0, x \to \pi \pm 0, x \to 3\pi/2 \pm 0, x \to 2\pi - 0$, the function (4.4) tends to $-\infty$ or $+\infty$, depending on ν.

The numerical analysis of $\varepsilon(x;\, \nu, \lambda)$ indicates that, in general, in the interval $[0, 2\pi)$ the graph of this function can have from two to six points of intersection with the straight line $\varepsilon = \text{const}$ depending on λ, ν and ε. Accordingly, the mechanical system under investigation can have from two to six relative states of equilibrium (with respect to the β coordinate).

In Figs 6 and 7 we show diagrams (similar to those in Figs 2, 3 and 5) to determine the number and type of stationary points of the function (4.2). Figure 6 corresponds to $\lambda = 0.75$ and $\nu = \pi/8$, while Fig. 7 corresponds to $\lambda = 0.75$ and $\nu = 5\pi/8$.

To conclude this section we will summarize the results on the motion of a rigid body or a two-degrees-of-freedom joint in a uniform gravitational field parallel to the fixed axis of the joint with respect to β. We will denote, by $\Pi_{\min}$ and $\Pi_{\max}$, respectively, the absolute minimum and maximum of $\Pi(\beta, L)$ with respect to β.

1. $\Pi(\beta, L)$ is a 2π-periodic function in β and, depending on the relationships between the projection of the angular momentum (L) onto the fixed axis of the joint and the inertial parameters of the rigid body, it can have from two to six stationary points in the range $0 \leqslant \beta < 2\pi$, which correspond to different states of equilibrium of the system with respect to β. Each stationary point of $\Pi(\beta, L)$ can be a maximum, a minimum, or a point of inflection. The minima correspond to stable states of equilibrium, while the maxima and the points of inflection correspond to unstable ones.

2. The motion of the system is possible only if the relationship between the first integrals (E and L) and the inertial parameters is such that $E \geqslant \Pi_{\min}$.

3. If $E > \Pi_{\max}$, the rigid body rotates periodically with respect to β. The period can be computed from (3.5).

4. If $\Pi_{\min} \leqslant E \leqslant \Pi_{\max}$ and E is not equal to $\Pi(\beta, L)$ at one of the stationary points, the rigid body oscillates periodically between β_- and β_+, where β_- and β_+ are two consecutive solutions of the equation $\Pi(\beta, L) = E$ such that $\Pi(\beta, L) < E$ for $\beta_- < \beta < \beta_+$. The period of these oscillations is finite and can be computed from (3.4). Note that several different oscillation intervals (β_-, β_+) may correspond to the same E and L.

5. If $\Pi_{\min} \leqslant E \leqslant \Pi_{\max}$ and at least one of the points β_- or β_+ defined in (4) coincides with a local maximum or point of inflection of $\Pi(\beta, L)$, the period of the corresponding oscillations of the body $\beta_- < \beta < \beta_+$ tends to infinity. A local maximum or point of inflection of $\Pi(\beta, L)$ determines the location of an unstable equilibrium with respect to β. It follows that steady motion such that $\beta = \beta_+$ or $\beta = \beta_-$ is possible in the case in question.

6. If $\Pi_{\min} \leqslant E \leqslant \Pi_{\max}$ and E coincides with one of the local minima of $\Pi(\beta, L)$, steady motion of the system is possible such that the rigid body is in one of the stable states of equilibrium with respect to β, which corresponds to the given local minimum.

The proof of points (2)–(6) follows from an analysis of the motion of a rigid body with respect to β as a conservative mechanical system with one degree of freedom for which the energy conservation equation has the form (2.4). Since the method of such an analysis has been presented in detail (see, for example, [2, 3]), we will state only the final results here. The motions (2)–(6) exhaust the set of qualitatively different motions of the system with respect to β.

The phase portraits of the system in the plane of β and $\dot\beta$ for various L can be constructed by solving Eqs (2.4) for β for all admissible E.

5. MOTION WITH RESPECT TO α WHEN $g_1 = g_2 = 0$

In the system under consideration, the motion with respect to α does not differ qualitatively from the inertial motion with respect to the same coordinate of a solid on a two-degrees-of-freedom joint [1]. All the relevant discussion and proofs contained in [1] can be carried over almost word-for-word to the case in hand. We shall therefore only state the final results.

In each period of rotation (oscillation) of the body with respect to β the angle α changes by the same amount $\Delta\alpha$. In the case of rotations

$$\Delta\alpha = L\Omega_r, \quad \Omega_r = \int_0^{2\pi} F(E, L, \beta)\, d\beta \tag{5.1}$$

$$F = \frac{1}{K(\beta)}\left[\frac{K(\beta) J_{11} - b^2(\beta)}{2EK(\beta) - L^2 - 2mgK(\beta)(\rho_2 \sin\beta + \rho_3 \cos\beta)}\right]^{1/2}$$

and in the case of oscillations

$$\Delta\alpha = L\Omega_\upsilon, \quad \Omega_\upsilon = 2\int_{\beta_-}^{\beta_+} F(E, L, \beta)\, d\beta \tag{5.2}$$

Here β_- and β_+ are the values of β corresponding to the extreme positions of the oscillating body ($\beta_- < \beta_+$). From (5.1) and (5.2) it follows that if $\mu_r = |L|\Omega_r/(2\pi)$ ($\mu_\upsilon = |L|\Omega_\upsilon/(2\pi)$) is a rational number, then the corresponding motion of the system as a whole is periodic, the shortest period being equal to $n_r\tau_r$ ($n_\upsilon\tau_\upsilon$), where $n_r(n_\upsilon)$ is the least natural denominator of the rational number $\mu_r(\mu_\upsilon)$ and $\tau_r(\tau_\upsilon)$ is the period of rotation (oscillation) of the body with respect to β. If $\mu_r(\mu_\upsilon)$ is an irrational number, the motion of the system is not periodic.

From (2.3) it follows that the state of equilibrium $\beta = \tilde\beta$ with respect to β corresponds to the rotation of the body about the fixed X_3 axis of the joint with constant angular velocity $\dot\alpha = L/K(\tilde\beta)$.

Formulae (5.1) and (5.2) can be obtained by integrating (2.3) with respect to time from t_0 to $t_0 + \tau_r$ ($t_0 + \tau_\upsilon$, respectively) with the variable integration replaced by β by virtue of (2.5). Here t_0 is an arbitrary initial instant of time. A detailed derivation of relationships similar to (5.1) and (5.2) is given in [1].

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